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ON CLOSURES OF ORBITS AND ARITHMETIC OF QUATERNIONS

BY

SHAHAR MOZES*

Department of Mathematics

University of Chicago, IL 60637, USA

ABSTRACT

For $G = \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$ we study the closures of orbits under the maximal split Cartan subgroup of G in homogeneous spaces $\Gamma\backslash G$. We show that if a closure of an orbit contains a closed orbit then the orbit is either dense or closed. We show the relation of this to divisibility properties of integral quaternions and other lattices.

In a recent series of papers M. Ratner (see [Ra]) proved the Raghunathan conjecture: If G is a real Lie group, $\Gamma < G$ a lattice and $H < G$ a subgroup generated by unipotent elements then for every $\Gamma x \in G$ we have $\overline{\Gamma x H} = \Gamma x F$ for some closed group F containing H. G.A. Margulis has conjectured in his ICM address, [Ma], that if A is the maximal $\mathbb{R}$-split Cartan subgroup of a semisimple real Lie group of rank ≥ 2, with no compact factors, and if $\Gamma x \in \Gamma\backslash G$ has a relatively compact A orbit then $\overline{\Gamma x A} = \Gamma x F$ for some closed subgroup $F < G$ (one has to put some conditions on Γ). A similar conjecture was made independently by H. Furstenberg.

Here we consider a special case of the analogous conjecture for groups over p-adic fields. Namely, we consider the case of the group

$$G = \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$$

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Current address: Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel.
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where p, ℓ are two fixed primes, $A = \left\{\left(\begin{pmatrix} * & \\ & * \end{pmatrix}, \begin{pmatrix} * & \\ & * \end{pmatrix}\right) \in G\right\}$ and $\Gamma < G$ an irreducible lattice (necessarily uniform). Then the conjecture has the following simple form:

CONJECTURE 1: *For G, Γ, A as above every A orbit, $\Gamma x A$ is either closed or dense.*

We prove a weaker result which says that non-closed orbits are dense if they satisfy some more conditions. The proof also gives some arithmetical corollaries concerning divisibility properties in Γ. We show that a strengthening of the arithmetical consequences is actually equivalent to Conjecture 1.

THEOREM 1: *Let p, ℓ be two primes $G = \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$, $\Gamma < G$ an irreducible (uniform) lattice, $A = \left\{\left(\begin{pmatrix} * & \\ & * \end{pmatrix}, \begin{pmatrix} * & \\ & * \end{pmatrix}\right) \in G\right\}$ a maximal split Cartan subgroup of G. Let $x \in \Gamma\backslash G$. Assume that $\overline{xA}$ contains a compact A orbit then xA is either dense or compact.*

There is no loss in generality in assuming that Γ is torsion free. By a result of A. Selberg [Sel] Γ contains a torsion free sublattice of finite index Γ' and the assertion for Γ follows from the one for Γ'. Henceforth we shall assume that Γ is torsion free.

The main tool we will use for proving this theorem is a "symbolic description" of the system $(\Gamma\backslash G, A)$. In [Mo1] we showed that a system $(\Gamma'\backslash G', A')$ where G' is a semisimple Chevalley group over a p-adic field, $\Gamma' < G'$ a lattice and $A' < G$ a split Cartan subgroup of G', is a compact group extension of a certain subshift of finite type. In the present case we obtain a two dimensional subshift of finite type $(\Omega, \mathbb{Z}^2)$. Let $M < A$ be the maximal compact subgroup of A. Note that $M = \left\{\left(\begin{pmatrix} r & \\ & r^{-1} \end{pmatrix}, \begin{pmatrix} s & \\ & s^{-1} \end{pmatrix}\right) \mid r \in \mathbb{Z}_p^*,\ s \in Z_\ell^*\right\}$, and that $M\backslash A \simeq \mathbb{Z}^2$. There exists a continuous map $\Phi: \Gamma\backslash G \to \Omega$ which induces a homeomorphism $\bar{\Phi}: \Gamma\backslash G/M \to \Omega$ conjugating the action of $M\backslash A \simeq \mathbb{Z}^2$ on $\Gamma\backslash G/M$ with $\mathbb{Z}^2$ action on Ω. Since M is a compact subgroup it follows that $\overline{xA} = \Phi^{-1}(\overline{\Phi(x)\mathbb{Z}^2})$. Hence in order to study the closures of A orbits in $\Gamma\backslash G$ it is enough to study the closures of $\mathbb{Z}^2$ orbits in Ω.

Description of Ω

Let $\triangle_p$, $\triangle_\ell$, $\triangle = \triangle_p \times \triangle_\ell$ denote the affine Bruhat-Tits buildings associated with $\mathrm{PGL}_2(\mathbb{Q}_p)$, $\mathrm{PGL}_2(\mathbb{Q}_\ell)$, $G = \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$ respectively (see

[B-T1,2], [Bro], [Ro] for general description of (affine) buildings and [Ser] for a description of the affine buildings associated with the groups $\mathrm{PGL}_2(k)$ where k is a nonarchimedean local field). Recall that Δ_p (resp. Δ_ℓ) is a $p+1$ (resp. $\ell+1$) regular tree. Let π_p: $\Delta \to \Delta_p$, π_ℓ: $\Delta \to \Delta_\ell$ be the corresponding projections. Since Γ is a torsion free lattice it acts freely on Δ. The quotient $Y = \Gamma\backslash\Delta$ is a two dimensional complex whose faces (cells), like those of Δ, are two dimensional squares. Note that the 1-skeleton of Δ is made of edges of two types:

(i) "Vertical edges" which project to an edge in Δ_p and a vertex in Δ_ℓ.

(ii) "Horizontal edges" which project to an edge in Δ_ℓ and a vertex in Δ_p.

Since the action of Γ preserves these types, the edges of Y may also be associated with these two types. Let $\mathcal{A} < \Delta$ be the apartment of Δ on which A acts by translations (via $M\backslash A \simeq \mathbb{Z}^2$). Let Ω : $\{\omega: \mathcal{A} \to Y |\ \omega \text{ admissible}\}$, where a map between two complexes is called **admissible** if it is locally an immersion and preserves the type (horizontal or vertical) of each edge. (Recall that $\Phi(\Gamma_g) = \pi \circ g|_{\mathcal{A}}$, see [Mo1].) It is convenient to think of the collection of vertices, edges and faces of Y as a finite set of colors (or labels). A map $\omega \in \Omega$ is a coloring (labelling) of the apartment $\mathcal{A}$ by these colors satisfying certain local conditions determined by combinatorial structure of Y. The covering map π: $\Delta \to Y$ gives a coloring of Δ. Notice that a face of Y is determined by its edges and actually already by any two non-parallel edges of it. It follows that an element $\omega \in \Omega$ is determined by its restriction to a pair of a vertical and a horizontal lines in $\mathcal{A}$.

We examine more closely the way the coloring of a horizontal and a vertical line determines the whole coloring. The roles of "vertical" and "horizontal", "p" and "ℓ" may be exchanged in what follows.

Fix a vertex $o = (o_p, o_\ell)$ in Δ. Let $\bar{o} = \pi(o)$ be its image in Y. Denote

$$\Gamma_p = \{\gamma \in \Gamma |\ \gamma o_\ell = o_\ell\},$$
$$\Gamma_\ell = \{\gamma \in \Gamma |\ \gamma o_p = o_p\}.$$

Let $R \subset \mathcal{A}$ be a rectangle with vertices a, b, c, d as in Figure 1.

Let φ: $R \to Y$ be an admissible map. Assume $\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = \bar{o}$. Hence the restriction of φ to each of the (directed) sides of R gives an element of $\pi_1(Y)$. Note that $\pi_1(Y)$ may be viewed as the group of deck transformations of π: $\Delta \to Y$ which is naturally identified with Γ. We have

$$\gamma_{ab} = \varphi|_{[a,b]} \in \Gamma_\ell, \quad \gamma_{cd} = \varphi|_{[c,d]} \in \Gamma_\ell, \quad \gamma_{ac} = \varphi|_{[a,c]} \in \Gamma_p, \quad \gamma_{bd} = \varphi|_{[b,d]} \in \Gamma_p.$$

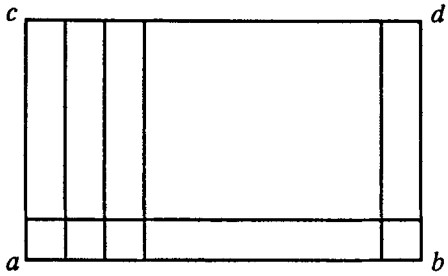

Figure 1

Since they bound a rectangle it follows that $\gamma = \gamma_{ab}\gamma_{bd} = \gamma_{ac}\gamma_{cd}$. Lift φ to $\tilde{\varphi}$: $R \to \triangle$ s.t. $\tilde{\varphi}(a) = o$. The vertex $\tilde{\varphi}(d)$ is the image of o under γ. As $o = (o_p, o_\ell)$ and $\gamma_{cd} = \Gamma_\ell$, we have

$$\gamma_{ab}\gamma_{bd}o_p = \gamma_{ac}\gamma_{cd}o_p = \gamma_{ac}o_p.$$

Notice that $\gamma_{bd}o_p = \pi_p \circ \tilde{\psi}_{bd}(d)$ where $\tilde{\psi}_{bd}$: $[b,d] \to \triangle$ is the lifting of $\psi_{bd} = \varphi|_{[b,d]}$: $[b,d] \to Y$ s.t. $\tilde{\psi}_{bd}(b) = o$. $\gamma_{ac}o_p = \pi_p \circ \tilde{\psi}_{ac}(c)$ where $\tilde{\psi}_{ac}$: $[a,c] \to \triangle$ is the lifting of $\psi_{ac} = \varphi|_{[a,c]}$: $[a,c] \to Y$ s.t. $\tilde{\psi}_{ac}(a) = o$. Thus we have shown:

LEMMA 1: *Let $R = [a,b,d,c] \subset \mathcal{A}$ be a rectangle, φ: $R \to Y$ an admissible map. Assume $\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = \bar{o}$. Let $\gamma_{ab} \in \Gamma_\ell \subset \Gamma$ be the element corresponding to $\varphi|_{[a,b]}$. Let ψ'_{ac}: $[a,c] \to \triangle_p$, ψ'_{bd}: $[b,d] \to \triangle_p$ be the two paths based at o_p, $\psi'_{ac} = \pi_p \circ \tilde{\psi}_{ac}$, $\psi'_{bd} = \pi_p \circ \tilde{\psi}_{bd}$ where $\tilde{\psi}_{ac}$: $[a,c] \to \triangle$, $\tilde{\psi}_{bd}$: $[b,d] \to \triangle$ are the liftings of $\varphi|_{[a,c]}$ and $\varphi|_{[b,d]}$ s.t. $\tilde{\psi}_{ac}(a) = o$, $\tilde{\psi}_{bd}(b) = o$, respectively. Then the path ψ'_{ac} is the image of the path ψ'_{bd} under γ_{ab}. Notice that ψ'_{ac}, ψ'_{bd} determines the coloring of the sides $[a,c]$, $[b,d]$ respectively.*

LEMMA 2:

(i) *Let $R' = [a,b,d',c'] \subset \mathcal{A}$ be a rectangle. Let φ': $R \to Y$ be an admissible map s.t. $\varphi'(a) = \varphi'(b) = \bar{o}$. Let $\gamma_{ab} \in \Gamma_\ell \subset \Gamma$ be the element corresponding to $\varphi'|_{[a,b]}$. Let $\psi'_{ac'}$: $[a,c'] \to \triangle_p$, $\psi'_{bd'}$: $[b,d'] \to \triangle_p$ be the two paths in $\triangle_p$ corresponding to the coloring of the sides of R based at o_p, as in Lemma 1. Then $\psi'_{ac'}$ is the image of $\psi'_{bd'}$ under γ_{ab}.*

(ii) *Let R'' be an infinite half strip as in Figure 2. Let φ'': $R'' \to Y$ be an admissible map s.t. $\varphi''(a) = \varphi''(b) = \bar{o}$. Let $\gamma_{ab} \in \Gamma_\ell \subset \Gamma$ be the element corresponding to $\varphi''|_{[a,b]}$. Let ψ''_a, ψ''_b be the paths in $\triangle_p$ based* *at o_p corresponding to the colorings of the sides of R'' above a and b respectively. Then ψ_a'' is the image of ψ_b'' under γ_{ab}.*

Proof: Assertion (ii) follows from (i) by looking at bigger and bigger subrectangles. Assertion (i) will follow from Lemma 1 if we can find a rectangle $R = [a,b,d,c] \supset R'$ and an admissible map φ: $R \to Y$ s.t. $\varphi|_{R'} = \varphi'$ and $\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = \bar{o}$. In [Mo1] we proved that the action of $\{(n,0)|\, n \in \mathbb{N}\}$ (the vertical shift) on Ω is topologically transitive (we remark here only that it follows from the fact that by the Howe–More theorem the system $(\Gamma\backslash G, \mathcal{B}, \mu, A)$ is mixing). Hence $\mathcal{C}_{\varphi'}^{R'} \cap \mathcal{C}_{\varphi'}^{R'}(n,0) \neq \Phi$ for some $n \in \mathbb{N}$ which may be chosen to be larger than the height of R'. Choose some $\omega \in \mathcal{C}_{\varphi'}^{R'} \cap T^n \mathcal{C}_{\varphi'}^{R'}$. Let $R = [a,b,d,c]$ be the rectangle in $\mathcal{A}$ containing R' of height n. Let $\varphi = \omega\,|_R$ then R, φ satisfies the required properties and (i) follows from Lemma 1. ∎

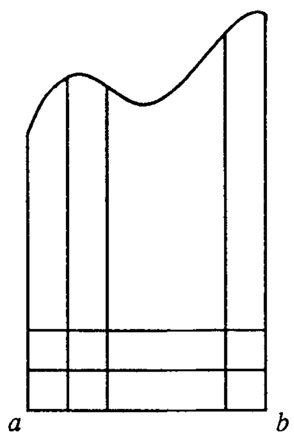

Figure 2

Lemma 3: *Let $\gamma = (\gamma_p, \gamma_\ell) \in \Gamma$ $\gamma \neq e$. Then both γ_p and γ_ℓ are nontrivial.*

Proof: Assume for example $\gamma_p = e$. Then $\gamma_\ell \neq e$. The projection of Γ in $\mathrm{PGL}_2(\mathbb{Q}_\ell)$ is dense since Γ is irreducible. It follows that the group generated by the conjugates of γ by the elements of Γ has a dense projection in $\mathrm{PGL}_2(\mathbb{Q}_\ell)$ while its projection in $\mathrm{PGL}_2(\mathbb{Q}_p)$ is the trivial element. As this group is contained in the discrete group Γ we have a contradiction. ∎

Lemma 4: *Let $x \in \Gamma\backslash G$ be s.t. xA is not compact and $\overline{xA} \ni y$ s.t. $yA = \overline{yA}$ is a compact orbit. Then there exists $z \in \overline{xA}$ such that there is a half plane $\mathcal{H} \subset \mathcal{A}$*

so that the corresponding tilings satisfy $\Phi(y)|_{\mathcal{H}} = \Phi(z)|_{\mathcal{H}}$ and $\Phi(y)|_E \neq \Phi(z)|_E$ where $E \in \mathcal{A}$ is an edge perpendicular to the boundary of $\mathcal{H}$ touching it and not contained in $\mathcal{H}$.

Proof: Recall that yA is a compact A orbit if and only if $\Phi(y)$ is a periodic coloring of $\mathcal{A}$. The assumption that $y \in \overline{xA}$ implies that $\Phi(y) \in \overline{\Phi(x)\mathbb{Z}^2}$. This means that we can find in $\Phi(x)$ larger and larger areas which are colored in the same way as $\Phi(y)$ up to translation. Notice that a maximal connected region where $\Phi(y)$ and a translate of $\Phi(x)$ have the same coloring is a rectangle, possibly stretching to infinity in some or all directions. This follows from the observation that the coloring of two perpendicular edges of a square determines the coloring of the other edges (see also [Mo1]). Since by the assumptions $\Phi(x) \neq \Phi(y)$ it follows using the compactness of the space Ω and the discreteness of the set of "colors" that we can find $\omega \in \overline{\Phi(x)\mathbb{Z}^2}$ which coincides with $\Phi(y)$ on a half plane $\mathcal{H}$ but not on the squares touching the boundary of $\mathcal{H}$. Let $z \in \overline{xA}$ be such that $\Phi(z) = \omega$. ∎

Fix $x \in \Gamma\backslash G$ s.t. xA is not compact and $\overline{xA}$ contains a compact A orbit $yA = \overline{yA}$. Let $z \in \overline{xA}$ be as in Lemma 4 and assume without loss of generality that $\Phi(z)$ and $\Phi(y)$ coincide on a half plane $\mathcal{H}$ bounded above by a horizontal line $\partial\mathcal{H}$.

LEMMA 5: *Let d be a horizontal period of $\Phi(y)$. Let $\{c_i |\, i \in \mathbb{N} \cup \{0\}\}$ be a sequence of points on $\partial\mathcal{H}$ at distance d from one another going to the left. The mapping $\Phi(z)\colon \mathcal{A} \to Y$ maps all of them to a point $\bar{o} = \pi(o)$, $o = (o_p, o_\ell) \in \Delta$. Let $\mathcal{L}_i$, $i \geq 0$ be a vertical ray in $\mathcal{A}$ going up from c_i. Let $\tilde{\varphi}_i\colon \mathcal{L}_i \to \Delta$ be the lifting of $\Phi(z)|_{\mathcal{L}_i}\colon \mathcal{L}_i \to Y$ s.t. $\tilde{\varphi}_i(c_i) = o$. Let $\psi_i = \pi_p \circ \tilde{\varphi}_i\colon \mathcal{L}_i \to \Delta_p$ be the corresponding path in Δ_p based at o_p. Identify each ψ_i with a point $[\psi_i] \in \partial\Delta_p$ in the boundary of the tree Δ_p. Then $\overline{\{[\psi_i] |\, i \geq 0\}}$ contains an open set.*

Proof: Let $\gamma \in \Gamma_\ell \subset G$ be the element corresponding to the map

$$\Phi(z)|_{[c_{i+1}, c_i]}\colon [c_{i+1}, c_i] \to Y.$$

(Notice that it doesn't depend on i.) By Lemma 2, $\psi_{i+1} = \gamma\psi_i$. Actually $\psi_{i+1} = \bar{\gamma}\psi_i$ where $\bar{\gamma}$ is the component of γ in $\mathrm{PGL}_2(\mathbb{Q}_p)$. We can by conjugating the whole structure assume that the stabilizer of o_p in $\mathrm{PGL}_2(\mathbb{Q}_p)$ is $\mathrm{PGL}_2(\mathbb{Z}_p)$. Thus $\bar{\gamma} \in \mathrm{PGL}_2(\mathbb{Z}_p)$, $\psi_i = \bar{\gamma}^i\psi_0$.

We shall use the following:

LEMMA 6: *Let $e \neq \bar\gamma \in \mathrm{PGL}_2(\mathbb{Z}_p)$ be of infinite order. Let ψ_0 be a geodesic path from $o_p \in \Delta_p$ to the boundary $\partial\Delta_p$. Assume that ψ_0 is not stabilized any power of $\bar\gamma$ then $\overline{\{\bar\gamma^n[\psi_0] \mid n \in \mathbb{N}\}} \subset \partial\Delta_p$ contains an open set.*

Proof: Since $\mathrm{PGL}_2(\mathbb{Z}_p)$ is a virtually pro-p group it follows that by replacing $\bar\gamma$ by some power of it we can assume that the image of $\bar\gamma$ under any homomorphism to a finite group has an order which is a power of p. This implies that if some power of $\bar\gamma$ stabilizes an edge in Δ_p then it acts on the p edges neighbouring to it at one vertex either as the identity permutation or as a full cycle of order p. Denote the vertices along the path ψ_0 by $(a_0, a_1, a_2, \dots)$. Replacing again $\bar\gamma$ by some power of it we can assume that $\bar\gamma$ stabilizes the first k edges of the path ψ_0, $k \geq 2$, and move the edge (a_k, a_{k+1}). We shall use the following:

CLAIM: Let $\alpha = \bar\gamma^r$ and $b_0 = o_p, b_1, b_2, \dots, b_m, b_{m+1}$, $m \geq 3$ be a geodesic path such that $\alpha b_i = b_i$ for $0 \leq i \leq m-1$ but $\alpha b_m \neq b_m$ then $\alpha^p b_m = b_m$ and α^p acts on the p edges from b_m different from (b_{m-1}, b_m) as a cycle for order p.

Proof of Claim: α being a power of $\bar\gamma$ acts on the p edges based at b_{m-1} different from (b_{m-2}, b_{m-1}) as a full cycle of order p. Hence $\alpha^p b_m = b_m$. Conjugating by an appropriate element of $\mathrm{PGL}_2(\mathbb{Q}_p)$ we can assume that $\mathrm{Stab}\, b_{m-2} = \mathrm{PGL}_2(\mathbb{Z}_p)$ and $b_m = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{PGL}_2(\mathbb{Z}_p)$, $b_{m+1} = \begin{pmatrix} p^3 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{PGL}_2(\mathbb{Z}_p)$ (recall that the vertices of Δ_p may be thought of as cosets of the maximal compact subgroup). α acts trivially on the $p+1$ vertices around b_{m-2} (since it stabilizes two of them it can't have a p-cycle there). It follows that there is a matrix $A \in GL_2(\mathbb{Z}_p)$ representing α so that $A = I + pB$, $B = (b_{ij}) \in M_2(\mathbb{Z}_p)$ and since $\alpha b_m \neq b_m$ we have $(I + pB)\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) \neq \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p)$ i.e.

$$\begin{pmatrix} p^{-2} & 0 \\ 0 & 1 \end{pmatrix} (I + pB) \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \notin \mathrm{PGL}_2(\mathbb{Z}_p) \Rightarrow p^{-1} b_{12} \notin \mathbb{Z}_p.$$

$A^p = (I + pB)^p = I + p^2 B + p^3 C$, for some $C \in M_2(\mathbb{Z}_p)$, corresponds to α^p. It follows that $A^p \begin{pmatrix} p^3 & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) \neq \begin{pmatrix} p^3 & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p)$, i.e. $\alpha^p b_{m+1} \neq b_{m+1}$. It follows that α^p acts on the p edges $\{(b_m, x) \mid x \neq b_{m-1}\}$ as a cycle of order p. ∎

Let $y_{k+1}, y_{k+2}, \dots, y_m$ be vertices s.t. $a_0, a_1, \dots, a_k, y_{k+1}, \dots, y_m$ is a geodesic path. By the action of an appropriate power $\bar\gamma^i$ $0 \leq i < p$ we can move the path ψ_0 to a path starting as $a_0 a_1 \dots a_k, y_{k+1}$. Then by the above claim we can act on it by some $\bar\gamma^{pj}$ so that its initial segment is $a_0, a_1, \dots, a_k, y_{k+1}, y_{k+2}$. Repeated use of the claim gives eventually a power $\bar{\gamma}^n$ which maps the path ψ_0 to a path $\bar{\gamma}^n\psi_0$ whose initial segment is $a_0, a_1, \ldots, y_{k+1}, \ldots, y_m$. Hence $\overline{\{\bar{\gamma}^n[\psi_0] | \, n \in \mathbb{N}\}}$ contains the open set of all the points in the boundary such that a geodesic path to them from o_p begins as $a_0, a_1, \ldots, a_k$. ∎

To complete the proof of Lemma 5 we have to verify that the path ψ_0 is not stabilized by any nontrivial power of $\bar{\gamma}$. By Lemma 3, $\bar{\gamma}^i \neq e$ for any $i \geq 1$. Since $\bar{\gamma}^i \in \mathrm{PGL}_2(\mathbb{Q}_p)$ it can stabilize at most two points in the boundary $\partial\Delta_p$. Let ξ_1, ξ_2 be the paths in Δ_p determined by the coloring induced by $\psi(y)$ of the two vertical rays going up and down based at the point c_0 (notation as in the statement of the lemma). Since $\psi(y)$ is periodic, it follows that both of them are stabilized by $\bar{\gamma}^i$. They are distinct because $\psi(y)$ is an admissible coloring. Since ψ_0 corresponds to a different point of the boundary $\partial\Delta_p$ it is not stabilized by $\bar{\gamma}^i$, $i \geq 1$. ∎

Proof of Theorem 1: If the orbit xA is closed there is nothing to prove. Assume that xA is not closed and that its closure contains a compact A-orbit yA. By Lemma 4 there exists an element $z \in \overline{xA}$ be as in the assumptions of Lemma 5. Lemma 5 asserts that $\Phi(z)$ colors a certain horizontal line $\mathcal{L}$ in $\mathcal{A}$ in a periodic pattern of period d and that if we look at the collection of vertical segments of any finite length based at a sequence of points $c_0, c_1, c_2, \ldots$ equispaced at distances sd from one another on $\mathcal{L}$ (for some fixed s) then $\Phi(z)$ colors them in all possible ways extending a fixed coloring of the first k edges. Let $\varphi\colon R \to Y$ be an admissible coloring of a $u \times v$ rectangle in $R \subset \mathcal{A}$, whose lower right corner can be assumed, w.l.o.g., to be c_0. Let $\mathcal{C}^R_\varphi$ be the corresponding cylindrical set in Ω. To show that xA is dense it is enough to show that some translate of $\Phi(z)$ lies in $\mathcal{C}^R_\varphi$. Let $M \subset \mathcal{A}$ be $k \times v$ rectangle such that its lower right corner is the point c_0, see Figure 3. Denote $\beta = \Phi(z)|_M$.

Let $\mathcal{C}^M_\beta$ be the corresponding cylindrical set. As mentioned earlier (see the proof of Lemma 2 and [Mo1]) the action of $\{(n,0) | \, n \in \mathbb{N}\}$ (the vertical shift) on Ω is topologically transitive. Hence there is some $n \in \mathbb{N}$ $n > k$ so that $\mathcal{C}^M_\beta \cap \mathcal{C}^R_\varphi(n,0) \neq \emptyset$. Fix some $\omega_o \in \mathcal{C}^M_\beta \cap \mathcal{C}^R_\varphi(n,0)$. Look at the coloring ω_0 induces on a vertical segment of length $u+n$ based at c_0 (see Figure 4). As observed at the beginning of the proof there exist some c_j $j \in \mathbb{N}$ such that $\Phi(z)$ induces the same coloring on the vertical segment of length $u+n$ based at c_j. Combined with the fact that the horizontal segment of length v to the left of

c_j is colored by $\Phi(z)$ in the same way as ω_0 colors the horizontal segment of length v to the left of c_0 and that these two colorings determines the coloring of a $(u+n) \times v$ rectangle it follows that there is a $u \times v$ rectangle in $\mathcal{A}$ which is colored by $\Phi(z)$ as φ colors R. ∎

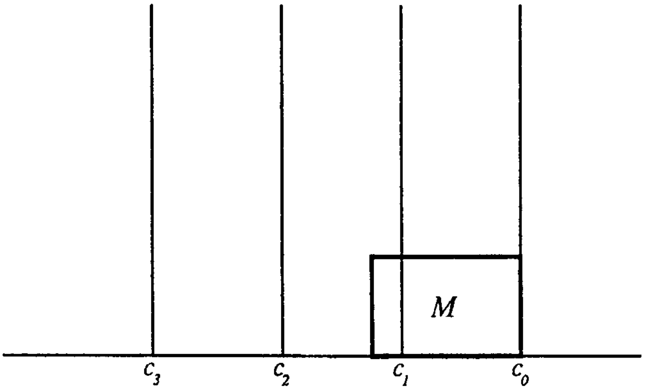

Figure 3

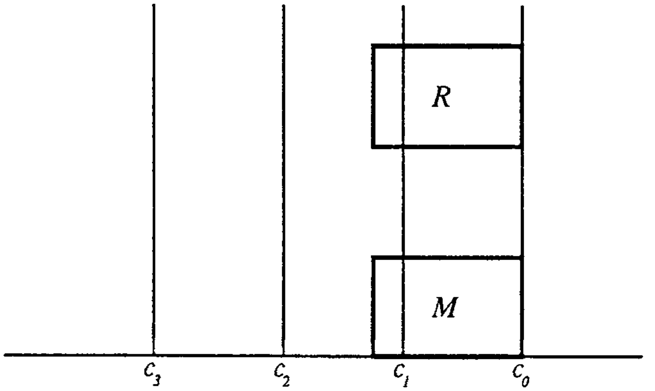

Figure 4

Divisibility

The lattice $\Gamma < G = \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$ acts on the building Δ of G. Assume that Γ is torsion free (hence the action is free). We want to define a notion of "divisibility" or "factorization" in Γ.

Definition:

(1) Fix a vertex $o \in \Delta$. We will say that $\gamma = \gamma'\gamma''$ is a factorization with respect to o if for every gallery $c_0, c_1, \ldots, c_d$ connecting o with

$\gamma' o$ and any gallery $D_0, D_1, \ldots, D_m$ connecting o with $\gamma'' o$, the sequence $c_0, c_1, \ldots, c_d, \gamma' D_0, \gamma' D_1, \ldots, \gamma' D_m$ may be completed (by adding chambers between c_d and $\gamma' D_0$) to form a gallery connecting o and γo. (*Remark*: In the present case of the building $\Delta = \Delta_p \times \Delta_\ell$ one needs only to add at most one more chamber (square), see Figure 5, however the definition makes sense also for groups acting on more general buildings where one may need to add more chambers.)

(2) We say that $x \in \Gamma$ divides $y \in \Gamma$ with respect to o if there exist $u, v \in \Gamma$ so that both ux and $(ux)v = y$ are factorizations.

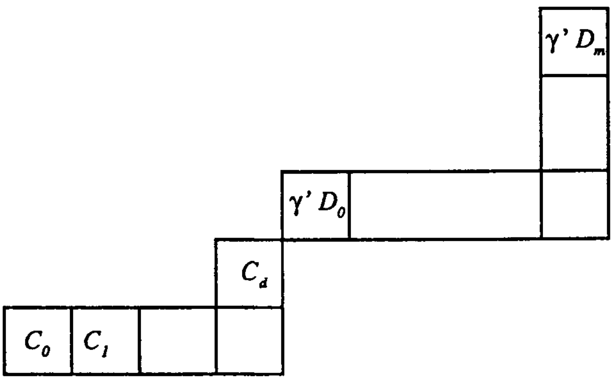

Figure 5

Remark: By abuse of notation we will usually just say "factorization" ("divides") instead of "factorization (divides) w.r.t. o". The motivation for this definition is the following example: Let $p, \ell \equiv 1 \pmod 4$ be two distinct primes. Let $\tilde{\Gamma} = \{x = x_0 + x_1 i + x_2 j + x_3 k \mid x \equiv 1 \pmod 2 \quad x_i \in \mathbb{Z} \quad |x|^2 = p^r \ell^s\}$. $\tilde{\Gamma}$ is a semigroup, which has natural notions of "factorization" and "divisibility". On the other hand $\tilde{\Gamma}$ may be used to construct an irreducible lattice $\Gamma < G = \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$ as follows: Let ξ_p, ξ_ℓ be roots of -1 in $\mathbb{Q}_p, \mathbb{Q}_\ell$ respectively. Define a map $\varphi: \tilde{\Gamma} \to G$ by

$$\tilde{\varphi}(x) = \left(\begin{pmatrix} x_0 + x_1\xi_p & x_2 + x_3\xi_p \\ -x_2 + x_3\xi_p & x_0 - x_1\xi_p \end{pmatrix}, \begin{pmatrix} x_0 + x_1\xi_\ell & x_2 + x_3\xi_\ell \\ -x_2 + x_3\xi_\ell & x_0 - x_1\xi_\ell \end{pmatrix} \right).$$

Let $\Gamma = \varphi(\tilde{\Gamma})$. (note that $\Gamma \simeq \tilde{\Gamma}/\{\pm p^r \ell^s\}$.) We claim that for this lattice the notions of factorization and division as defined above coincide with the natural ones. The verification of this involves identifying the 1-skeleton of Δ with the Cayley graph of Γ with respect to a (natural) set of generators which are the images under φ of the "prime" factors (generators) of $\tilde{\Gamma}$. For more details of this identification we refer to [L], [Mo1], [Mo2]. ∎

We return to the general case. Let $o = (o_p, o_\ell) \in \triangle$ $\bar{o} = \pi(o)$ be fixed as above. There is a one to one correspondence between elements of Γ and admissible maps $\varphi\colon R \to Y$ of finite rectangles $R = [a,b,d,c] \subset \mathcal{A}$ (which may be degenerate, i.e. of 0 width or height) such that $\varphi(a) = \bar{o} = \varphi(d)$. Given such a map $\varphi\colon R \to Y$ there exists a unique lifting $\tilde{\varphi}\colon R \to \triangle$ of φ such that $\varphi(a) = o$. The element $F(\varphi) \in \Gamma$, corresponding to φ is the unique element of Γ mapping o to $\varphi(d)$. Conversely given $\gamma \in \Gamma$ let $\mathrm{Conv}[o, \gamma o] \subset \triangle$ be the convex hull of $o, \gamma o$, i.e. the intersection of all the apartments containing o and γo. $\mathrm{Conv}[o, \gamma o]$ is isometric to a rectangle $R \subset \mathcal{A}$ and we obtain an admissible map $F^{-1}(\gamma)\colon R \to Y$ whose lift to $\triangle$ based at o maps R to $\mathrm{Conv}[o, \gamma o]$.

LEMMA 7: *Let $\gamma_1, \gamma_2 \in \Gamma$ γ_1 divides γ_2 if and only if the corresponding admissible maps $\varphi_1 = F^{-1}(\gamma_1)\colon R_1 \to Y$, $\varphi_2 = F^{-1}(\gamma_2)\colon R_2 \to Y$ are such that there is a rectangle $R_1' \subset R_2$ isometric to R_1 so that $\varphi_2|_{R_1'}$ coincides with φ_1 when we identify R_1' and R_1 (in other words : "φ_1 appears in φ_2").*

Proof: Assume there exists a rectangle $R_1' \subset R_2$ as in the lemma. Let the rectangles be $R_2 = [a,b,d,c]$ $R_1' = [e,f,h,g]$ (see Figure 6).

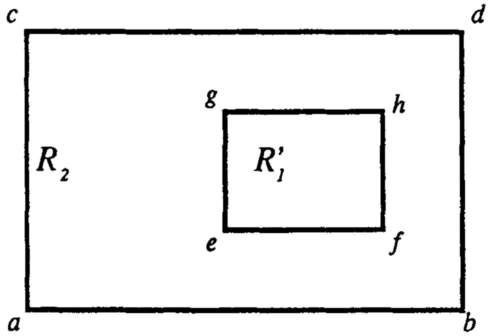

Figure 6

Let $\gamma' \in \Gamma$ be the element corresponding to φ_2 restricted to the rectangle whose bottom left corner is a and top right corner is e. Let $\gamma'' \in \Gamma$ be the element corresponding to the rectangle whose bottom left corner is h and top right corner is d. It follows that $\gamma_2 = \gamma'\gamma_1\gamma''$ is a factorization. Conversely assume γ_1 divides γ_2, i.e. $\exists \gamma', \gamma'' \in \Gamma$ s.t. $\gamma_2 = \gamma'\gamma_1\gamma''$ is a factorization. It follows that the vertices $o, \gamma' o, \gamma'\gamma_1 o, \gamma'\gamma_1\gamma'' o = \gamma_2 o$ lie along a gallery connecting o and $\gamma_2 o$. Hence the rectangle $\operatorname{Conv}[\gamma' o, \gamma'\gamma_1 o]$ is contained in the rectangle $\operatorname{Conv}[o, \gamma_2 o]$, moreover it is the image under γ' of the rectangle $\operatorname{Conv}[o, \gamma_1 o]$. It follows that the maps $\varphi_1 = F^{-1}(\gamma_1)\colon R_1 \to Y$, $\varphi_2 = F^{-1}(\gamma_2)\colon R_2 \to Y$, are as in the lemma. ∎

A subshift Σ of Ω (i.e., a closed shift invariant subset) is characterized by the collection of all the finite "windows" one sees in the elements of Σ. By this we mean the restrictions of any element of Ω to any finite rectangle ignoring the position (up to translation) of the rectangle. Conversely a collection of arbitrarily large windows, defines a subshift of Ω consisting of the elements $\omega \in \Omega$ so that the restriction of ω to any finite rectangle appears in one of the given windows.

LEMMA 8: *Let $\mathcal{C} = \{\gamma_i \mid i \in \mathbb{N}\}$ be given. Associated with each γ_i is an admissible map $F^{-1}(\gamma_i)\colon R_i \to Y$ ("a window"). Assume that the rectangles R_i become arbitrarily large (in both dimensions). Let $\Sigma = \Sigma(\mathcal{C}) \subset \Omega$ be the subshift corresponding to this collection of windows. Then $\Sigma = \Omega$ if and only if for every $\gamma \in \Gamma$ there exists some $\gamma_i \in \mathcal{C}$ such that γ divides γ_i.*

Proof: If $\Sigma = \Omega$ then for every $\gamma \in \Gamma$ there exists some $\gamma_i \in \mathcal{C}$ such that $F^{-1}(\gamma)$ appears in $F^{-1}(\gamma_i)$. By Lemma 7 this implies that γ divides γ_i. Conversely if for every $\gamma \in \Gamma$ there exist a $\gamma_i \in \mathcal{C}$ divisible by it then every admissible map $\varphi\colon R \to Y$ of a rectangle R appears in some $F^{-1}(\gamma_i)$ and $\Omega = \Sigma$. ∎

From Theorem 1, or rather its proof, it follows that

COROLLARY 1: *If $\mathcal{C} = \{\gamma_i \mid i \in \mathbb{N}\}$ is such that there exists an admissible map $\varphi\colon Q \to Y$ of a quadrant Q of $\mathcal{A}$ such that any finite window in φ appears in some $F^{-1}(\gamma_i)$ and such that, w.l.o.g., the restriction of φ the to horizontal boundary is periodic and the whole map is not. Then for every $\gamma \in \Gamma$ there is some $\gamma_i \in \mathcal{C}$ divisible by γ.*

For $\gamma \in \Gamma$ let $\|\gamma\|_o = \operatorname{Dist}(o, \gamma o)$ where the distance is measured in the 1-skeleton graph of Δ. Let $\|\gamma\|_{o_p} = \operatorname{Dist}(o_p, \gamma o_p)$, $\|\gamma\|_{o_\ell} = \operatorname{Dist}(o_\ell, \gamma o_\ell)$ where the distances are measured in the corresponding trees. Notice that $\|\gamma\|_o = \|\gamma\|_{o_p} + \|\gamma\|_{o_\ell}$ and that $\gamma = \gamma_1\gamma_2$ is a factorization (w.r.t. o) iff $\|\gamma\|_o = \|\gamma_1\|_o + \|\gamma_2\|_o$.

COROLLARY 2: *Let $e \neq \alpha \in \Gamma$ and $\gamma_n \in \Gamma$ $n \in \mathbb{N}$ be such that*

(1) $\|\alpha^n\gamma_n\|_{o_p} \to \infty$, $\|\alpha^n\gamma_n\|_{o_\ell} \to \infty$,
(2) $\|\alpha^n\gamma_n\|_o \geq \|\alpha^n\|_o + \|\gamma_n\|_o - C$ *for some constant C (bounded cancellation).*

Then either the subshift determined by the collection $\{F^{-1}(\alpha^n\gamma_n)\}$ consists only of one (up to symmetry) periodic orbit (which is uniquely determined by α), or for any $x \in \Gamma$ there is $n \in \mathbb{N}$ so that x divides $\alpha^n\gamma_n$.

Proof: We have associated with any $\beta \in \Gamma$ a map $F^{-1}(\beta)$ from a finite rectangle in $\mathcal{A}$ to Y. Consider the maps $F^{-1}(\alpha^n)$, since $\alpha \neq e$ and Γ is torsion free it follows that as n becomes large the restriction of $F^{-1}(\alpha^n)$ to some segments is a longer and longer repetition of some fixed pattern. Assumption 2 guarantees that there is little cancelation when we multiply α^n by γ_n. Hence as n tends to ∞ there are longer and longer segments (horizontal or vertical) in $\mathcal{A}$ such that the restriction of $F^{-1}(\alpha^n\gamma_n)$ to them is a long repetition of some fixed pattern ρ: $I \to Y$. Together with assumption 1 this implies that either we can find an admissible map φ: $Q \to Y$ of a quadrant as in Corollary 1 which implies that for every $x \in \Gamma$ there exists some $\alpha^n\gamma_n$ divisible by x or that the subshift Σ corresponding to $\{F^{-1}(\alpha^n\gamma_n)|\ n \in \mathbb{N}\}$ consists of one periodic point. To see the uniqueness of this periodic orbit observe that if $\sigma \in \Sigma$ then there is a line $\mathcal{L} \subset \mathcal{A}$ so that $\sigma|_{\mathcal{L}}$: $\mathcal{L} \to Y$ is made of infinite repetition of the map ρ. Assume w.l.o.g. $\mathcal{L}$ is horizontal. Let $\mathcal{M}$ be a vertical line in $\mathcal{A}$ intersecting $\mathcal{L}$ at a vertex Z. Lift $\sigma|_{\mathcal{M}}$: $\mathcal{M} \to Y$ to $\tilde{\sigma}|_{\mathcal{M}}$: $\mathcal{M} \to \Delta$ so that $\sigma(Z) = o'$. Let $\theta \in \Gamma_\ell$ be the element of γ corresponding to the period of σ in the direction of $\mathcal{L}$ based at Z (which by choosing $\mathcal{L}$ and Z can be assumed to be ρ). It follows that (see Lemma 2) $\tilde{\sigma}|_{\mathcal{M}}(\mathcal{M})$ is an infinite line in Δ_p stabilized pointwise by θ. Since $\theta \neq e$ this line is uniquely determined by θ, hence by α, and it follows that the periodic point is determined by α (up to (reflection) symmetry). ∎

To illustrate the arithmetical meaning of this corollary we state a special case of it for the semigroup of quaternions mentioned above:

COROLLARY 3: *Let $x = x_0 + x_1 i + x_2 j + x_3 k$, $y = y_0 + y_1 i + y_2 j + y_3 k$ be integral quaternions s.t. x_0, x_1, x_2, x_3 as well as y_0, y_1, y_2, y_3 are relatively prime $|x|^2 = p^r$ $|y|^2 = \ell^s$ $r, s \geq 1$. Then either $xy = \pm yx$ or for any quaternion $z = z_0 + z_1 i + z_2 j + z_3 k$ s.t. z_0, z_1, z_2, z_3 are relatively prime and $|z|^2 = p^k \ell^m$. There exists some n s.t. z divides $x^n y^n$, i.e., $x^n y^n = uzv$ for some integral quaternions u, v.*

We conclude by remarking that Conjecture 1 is actually equivalent to the following strengthening of Corollary 2.

CONJECTURE 2: *Let $\gamma_n \in \Gamma$ $n \in \mathbb{N}$ be such that $\|\gamma_n\|_{o_p}, \|\gamma_n\|_{o_\ell} \to \infty$ then either every $x \in \Gamma$ divides some γ_n or there are finitely many rank 2 abelian subgroups of Γ and finitely many cosets of them which contain $\{\gamma_n |\, n \in \mathbb{N}\}$.*

Proof of the equivalence: The preceding discussion shows that every $x \in \Gamma$ divides some γ_n iff the subshift Σ corresponding to $\{F^{-1}(\gamma_n)\}$ is equal to Ω. Conjecture 1 says that $\Sigma = \Omega$ unless it contains only periodic points. It is easily verified that a closed shift invariant set containing only periodic points must be finite. Hence conjecture 1 implies that if there is $x \in \Gamma$ not dividing any γ_n then the subshift Σ defined by $\{F^{-1}(\gamma_n)\}$ consists of finitely many periodic points. This implies that $\{\gamma_n |\, n \in \mathbb{N}\}$ is contained in a union of finitely many cosets of rank 2 abelian groups. (Notice that each periodic point defines finitely many, conjugate, rank 2 abelian subgroups of Γ, see [Mo1].) To see the opposite implication, let $\omega \in \Omega$ be non periodic. Fix $\bar{o} \in Y$ a vertex which has infinitely many preimages, under the map ω: $\mathcal{A} \to Y$, in $\mathcal{A}$ along some diagonal ray. Denote these, ordered, by $a_0, a_1, a_2, \ldots$ and let $\gamma_n = F^{-1}(\omega|_{\mathrm{Conv}[a_0,a_n]})$ be the element of Γ corresponding to the rectangle with opposite corners a_0, a_n. Since ω is non periodic, the diagonal ray may be chosen so that $\{\gamma_n |\, n \in \mathbb{N}\}$ are not contained a finite union of cosets of abelian subgroups and $\|\gamma_n\|_p, \|\gamma_n\|_\ell \to \infty$. Hence by Conjecture 2 it follows that every $x \in \Gamma$ divides some γ_n which implies that ω has a dense orbit in Ω. ∎

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